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Second order homological obstructions on real algebraic manifolds

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Abstract

Let Y be a compact nonsingular real algebraic set whose homology classes (over $\mathbb{Z}/2$) are represented by Zariski closed subsets. It is well known that every smooth map from a compact smooth manifold to Y is unoriented bordant to a regular map. In this paper, we show how to construct smooth maps from compact nonsingular real algebraic sets to Y not homotopic to any regular map starting from a nonzero homology class of Y of positive degree. We use these maps to obtain obstructions to the existence of local algebraic tubular neighborhoods of algebraic submanifolds of $\mathbb{R}^n$ and to study some algebro-homological properties of rational real algebraic manifolds.

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1. Introduction and main results

Let Y be a real algebraic manifold, i.e., a nonsingular real algebraic set. Suppose Y compact. A homology class of Y (over $\mathbb{Z}/2$) is called algebraic if it is represented by a Zariski closed subset of Y. Let $m := \dim(Y)$ and let $k \in \{0, 1, \dots, m\}$. Indicate by $H_k^{alg}(Y, \mathbb{Z}/2)$ the subgroup of $H_k(Y, \mathbb{Z}/2)$ consisting of all algebraic homology classes of Y of degree k. If $H_k(Y, \mathbb{Z}/2) = H_k^{alg}(Y, \mathbb{Z}/2)$, then $H_k(Y, \mathbb{Z}/2)$ is said to be algebraic. Define the algebraic homology group $H_*^{alg}(Y, \mathbb{Z}/2)$ of Y by $H_*^{alg}(Y, \mathbb{Z}/2) := \bigoplus_{k=0}^{m} H_k^{alg}(Y, \mathbb{Z}/2)$. This group plays a crucial role in the study of the classical problem of making smooth objects algebraic (see Chapters 11–14 of [3]). Let us recall two of the main aspects of this fact. Let N be a compact smooth submanifold of Y and let $[N]$ be the homology class of Y represented by N. In order to approximate N in Y by algebraic submanifolds, the condition $[N] \in H_*^{alg}(Y, \mathbb{Z}/2)$ is necessary. There are cases in which this condition is sufficient also: for example, when N has codimension one or when Y is a 3-fold and N is a curve (see [5] and Section 12.4 of [3]). Examples of submanifolds N which do not verify the preceding necessary condition can be obtained as follows. By a famous result of Thom [18, Thm. II.26], we know that, if $n \leqslant \frac{1}{2}m$ or $n = m - 1$, each homology class in $H_n(Y, \mathbb{Z}/2)$ is realizable by a smooth submanifold of Y. In this

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way, for each non-algebraic element of $H_n(Y,\mathbb{Z}/2)$, there exists a n-dimensional compact smooth submanifold N of Y which is not approximable in Y by algebraic submanifolds. Consider now a compact real algebraic manifold X and a smooth map $f: X \to Y$. In the real algebraic setting, it is natural to wonder whether f has at least one of the following algebraic properties:

(1) *the unoriented bordism class of f is algebraic, i.e., there exist a compact smooth manifold W with boundary and a continuous map $F: W \to Y$ such that $\partial W = X \sqcup X'$ where X' is a compact real algebraic manifold, $F|_X = f$ and $F|_{X'}$ is a regular map* (see [7, pp. 18–19]);
(2) *f is homotopic to a regular map*,

where, evidently, (2) implies (1). Property (1) is closely related to the algebraic properties of the homology of Y. In fact, the truthfulness of each of the preceding properties is subordinated to the fact that f maps the fundamental homology class $[X]$ of X into $H_*^{alg}(Y,\mathbb{Z}/2)$. By the Steenrod representability theorem [18] and Tognoli's theorem [19], it follows that, for each homology class α of Y, there exists a smooth map $f: X \to Y$ from a compact real algebraic manifold X to Y such that $f_*([X]) = \alpha$. If α is not algebraic, then f cannot satisfy property (1). In particular, if the unoriented bordism class of every smooth map from a compact real algebraic manifold to Y is algebraic, then the homology of Y is totally algebraic, i.e., $H_*(Y,\mathbb{Z}/2) = H_*^{alg}(Y,\mathbb{Z}/2)$. The converse of the latter fact is also true. It follows from remarkable results of Differential Topology (see Lemma 2.7.1 of [1]).

Theorem. *(See Thom [18], Milnor [15], Conner and Floyd [7].) If Y has totally algebraic homology, then the unoriented bordism class of every smooth map from a compact real algebraic manifold to Y is algebraic.*

The previous arguments show how the existence of non-algebraic homology classes of Y generates obstructions to the possibility of realizing algebraic properties of smooth objects defined on Y. Because of their importance, we call the obstructions induced by the inequality $H_*(Y,\mathbb{Z}/2) \neq H_*^{alg}(Y,\mathbb{Z}/2)$ *first order obstructions on Y*. In literature, there are many examples of compact real algebraic manifolds without totally algebraic homology (see Section 11.3 of [3] and related references). At the moment, among the first order obstructions, the following is certainly the deepest.

Theorem. *(See Benedetti and Dedò [2], Teichner [17].) For each integer $m \geqslant 6$, there exist a compact connected real algebraic manifold Y of dimension m, a compact connected real algebraic manifold X of dimension $m-2$ and a smooth map $f: X \to Y$ such that, for every homeomorphism $h: Y \to Y'$ from Y to a real algebraic manifold Y', the unoriented bordism class of $h \circ f$ is not algebraic.*

This paper deals with the construction of smooth maps $f: X \to Y$ between compact real algebraic manifolds not homotopic to any regular map when the target space Y has totally algebraic homology, i.e., when the first order obstructions on Y do not occur. In this situation, we also discover that, if there exists a nonzero homology class of Y of positive degree, then it is possible to construct such maps. We call this kind of homological obstructions as *second order obstructions on Y*. Our first obstruction result is as follows (see Remark 2.1 also).

Theorem 1.1. *Let Y be a real algebraic manifold of dimension m, let K be a compact semialgebraic subset of Y of dimension $k > 0$ and let U be a neighborhood of K in Y. Suppose that the homology class $[K] \in H_k(Y,\mathbb{Z}/2)$ represented by K is nonzero. Then there exist a $(k+1)$-dimensional compact real algebraic manifold X and a smooth map $f: X \to Y$ with the following properties: $f(X) \subset U$ and, for every homotopy equivalence $h: Y \to Y'$ from Y to a m'-dimensional compact real algebraic manifold Y' with $H_{m'-k}(Y',\mathbb{Z}/2)$ algebraic, the map $h \circ f$ is not homotopic to any regular map.*

As a consequence, we obtain the following result which improves Theorem 1 of [9] and, in some sense, extends the Benedetti–Dedò–Teichner theorem to the setting of second order obstructions.

Corollary 1.2. *Let Y be a real algebraic manifold of dimension m. Suppose $H_k(Y,\mathbb{Z}/2) \neq \{0\}$ for some positive integer k. Then there exist a $(k+1)$-dimensional compact real algebraic manifold X and a smooth map $f: X \to Y$*

with the following property: for every homotopy equivalence $h: Y \to Y'$ from Y to a compact real algebraic manifold Y' with totally algebraic homology, the map $h \circ f$ is not homotopic to any regular map.

In the remainder of this section, we present some applications of Theorem 1.1 concerning both the existence of algebraic tubular neighborhoods and the rationality of algebraic submanifolds of $\mathbb{R}^n$.

Let Y be an algebraic submanifold of $\mathbb{R}^n$. Given a subset S of Y, we say that Y has an *algebraic tubular neighborhood locally at S in $\mathbb{R}^n$* (a *ATN locally at S* for short) if there exist a Zariski open neighborhood Ω of S in $\mathbb{R}^n$ and a regular map $\rho: \Omega \to Y$ such that $\rho(y) = y$ for each $y \in \Omega \cap Y$. If Y has a ATN locally at some of its nonempty subsets, then Y is said to have a ATN. Combining Theorem 1.1 with the Stone–Weierstrass approximation theorem, we infer:

Theorem 1.3. *Let Y be a compact algebraic submanifold of $\mathbb{R}^n$ of dimension m and let K be a compact semialgebraic subset of Y of dimension $k > 0$. Suppose $[K] \neq 0$ and $H_{m-k}(Y, \mathbb{Z}/2)$ algebraic. Then Y does not have any ATN locally at K.*

In particular, we have:

Corollary 1.4. *(See [9], Thm. 2.) For each nonnegative integer n, the unique compact algebraic submanifold Y of $\mathbb{R}^n$ which has a global ATN (i.e., a ATN locally at Y) is the single point.*

Recall that an irreducible real algebraic manifold is said to be rational if it has a nonempty Zariski open subset biregularly isomorphic to a Zariski open subset of some $\mathbb{R}^m$. The existence of algebraic tubular neighborhoods of $Y \subset \mathbb{R}^n$ is closely related to the rationality of Y itself. In fact, the following elementary result holds.

Lemma 1.5. *Let Y be an irreducible algebraic submanifold of $\mathbb{R}^n$ of dimension m.*
We have:

(1) *If Y has a ATN, then it is unirational, i.e., there exists a dominating regular map from a Zariski open subset of $\mathbb{R}^m$ to Y.*
(2) *If Y is rational, then it has a ATN. More precisely, if Z is a nonempty Zariski open subset of Y biregularly isomorphic to a Zariski open subset of $\mathbb{R}^m$, then Y has a ATN locally at Z.*

Thanks to Theorem 2.4 of [13, p. 170] and Corollary 6.5 of [16, p. 137], we know that every unirational connected real algebraic manifold of dimension $\leqslant 2$ is rational. This fact and Lemma 1.5 imply the following:

Corollary 1.6. *Let Y be a connected algebraic submanifold of $\mathbb{R}^n$ of dimension $\leqslant 2$. Then Y has a ATN if and only if it is rational.*

Remark 1.7. The following result is an immediate consequence of Theorems 1.9 and 1.12 of [12] or, equivalently, of Lemma 3 and Theorem 5(a) of [10]: Let Y be an algebraic submanifold of $\mathbb{R}^n$ of positive dimension. Then there exist an algebraic submanifold $\widetilde{Y}$ of $\mathbb{R}^{2n}$ and a regular map $\varphi: \widetilde{Y} \to Y$ such that φ is a Nash isomorphism and $\widetilde{Y}$ does not have any ATN.

Theorem 1.3 and Lemma 1.5(2) generate some algebro-homological obstructions for a Zariski open subset of a rational compact real algebraic manifold to be biregularly isomorphic to a Zariski open subset of some $\mathbb{R}^m$:

Theorem 1.8. *Let Y be a rational compact real algebraic manifold of dimension m and let Z be a nonempty Zariski open subset of Y biregularly isomorphic to a Zariski open subset of $\mathbb{R}^m$. Then, for each positive integer k such that $H_{m-k}(Y, \mathbb{Z}/2)$ is algebraic, the homomorphism $i_k^*: H_k(Z, \mathbb{Z}/2) \to H_k(Y, \mathbb{Z}/2)$ induced by the inclusion $Z \hookrightarrow Y$ is null. In particular, i_m^* (resp., i_{m-1}^*) is null if $m \geqslant 1$ (resp., $m \geqslant 2$).*

Remark 1.9. The reader observes that the preceding result is false if, in the statement, we replace "biregularly isomorphic" with "Nash isomorphic". Let us give a counterexample. Let $S^1 \subset \mathbb{R}^2$ be the standard circle and let $p \in S^1$. Define $Y := S^1 \times S^1 \subset \mathbb{R}^4$ and $Z := S^1 \times (S^1 \setminus \{p\})$. It is easy to verify that Z is Nash isomorphic to $\mathbb{R}^2 \setminus \{0\}$, but $i_1^* : H_1(Z, \mathbb{Z}/2) \to H_1(Y, \mathbb{Z}/2)$ is not null.

Preprint [8] deals with Theorem 1.3 and a slightly different version of Theorem 1.1. Preprint [11] contains some variants and a complete proof of Theorem 1.1. By using different techniques, Bochnak and Kucharz have independently proved some results similar to Theorem 1.1 (see [6]). We wish to thank R. Benedetti, M. Shiota and A. Tognoli for several useful discussions.

2. Proofs

Proof of Theorem 1.1. By Théorème III.2 of [18], there exist a k-dimensional compact smooth manifold A' and a smooth map $\varphi' : A' \to Y$ such that $\varphi'(A') \subset U$ and $(\varphi')_*([A']) = [K]$. Since $[K] \neq 0$, we can find a connected component A of A' such that $(\varphi')_*([A]) \neq 0$. Define the map $\varphi : A \to Y$ as the restriction of φ' to A. By Tognoli's theorem [19], we may suppose that A is a compact real algebraic manifold. Let W be an irreducible real algebraic manifold having two connected components W_1 and W_2 both diffeomorphic to the standard circle S^1, let $\psi : S^1 \to W_1$ be a diffeomorphism and let $j : W_1 \hookrightarrow W$ be the inclusion map. Let $\pi_1 : S^1 \times A \to S^1$ and $\pi_2 : S^1 \times A \to A$ be the natural projections. Applying Theorem 2.8.3 of [1] to $j \circ \psi \circ \pi_1$ (see Lemma 4 of [9] also), we obtain a compact real algebraic manifold X, a connected component X_0 of X, a diffeomorphism $\pi : X_0 \to S^1 \times A$ and a regular map $R : X \to W$ such that $R|_{X_0}$ is arbitrarily close to $j \circ \psi \circ \pi_1 \circ \pi$ in $C^\infty(X_0, W)$, equipped with the C^∞-topology. Fix a point $y \in K$ and define $f : X \to Y$ by $f := \varphi \circ \pi_2 \circ \pi$ on X_0 and $f(x) := y$ for each $x \in X \setminus X_0$. Let $h : Y \to Y'$ be a homotopy equivalence from Y to a m'-dimensional compact real algebraic manifold Y' with $H_{m'-k}(Y', \mathbb{Z}/2)$ algebraic. Indicate by $D : H^*(Y', \mathbb{Z}/2) \to H_*(Y', \mathbb{Z}/2)$ the Poincaré duality isomorphism. Since $(h \circ \varphi)_*([A])$ is a nonzero element of $H_k(Y', \mathbb{Z}/2)$ and $H_{m'-k}(Y', \mathbb{Z}/2)$ is algebraic, there exists a $(m'-k)$-dimensional Zariski closed subset Z' of Y' such that $D^{-1}((h \circ \varphi)_*([A])) \cup D^{-1}([Z']) \neq 0$. Indicate by $\mathrm{Nonsing}(Z')$ the set of all nonsingular points of Z' of dimension $m' - k$. By the Thom transversality theorem, there exists a smooth map $g : A \to Y'$ homotopic to $h \circ \varphi$ such that $g(A) \cap Z' \subset \mathrm{Nonsing}(Z')$ and g is transverse to $\mathrm{Nonsing}(Z')$ in Y'. In this way, it holds: $0 \neq D(D^{-1}(g_*([A])) \cup D^{-1}([Z'])) = g_*([g^{-1}(Z')])$. It follows that $[g^{-1}(Z')] \neq 0$ and hence $g^{-1}(Z')$ is a finite subset of A formed by an odd number of points. Choose $y' \in Y' \setminus Z'$ arbitrarily close to $h(y)$ in Y'. Define $f' : X \to Y'$ by $f' := g \circ \pi_2 \circ \pi$ on X_0 and $f'(x) := y'$ for each $x \in X \setminus X_0$. Observe that f' is homotopic to $h \circ f$. We will prove that f' is not homotopic to any regular map completing the proof. Suppose on the contrary that this is false. Applying Proposition 2.8.8 of [1] to f', we obtain a compact real algebraic manifold T, a diffeomorphism $\xi : T \to X$ and a regular map $P : T \to Y'$ such that ξ is a regular map and P is arbitrarily close to $f' \circ \xi$ in $C^\infty(T, Y')$. Define $B' := \xi^{-1}\pi^{-1}(S^1 \times g^{-1}(Z'))$ and $B'' := P^{-1}(Z')$. From the definition of f', it follows that $B' = (f' \circ \xi)^{-1}(Z')$ and $f' \circ \xi$ is transverse to $\mathrm{Nonsing}(Z')$ in Y'. Choosing P sufficiently close to $f' \circ \xi$, we have that B'' is a compact real algebraic submanifold of T and, thanks to Theorem 14.1.1 of [3], there exists a smooth embedding η of B'' in T arbitrarily close to the inclusion map $B'' \hookrightarrow T$ in $C^\infty(B'', T)$ such that $\eta(B'') = B'$. Let $\alpha : B'' \to W$ be the smooth submersion defined by $\alpha := j \circ \psi \circ \pi_1 \circ \pi \circ \xi \circ \eta$. Observe that $\alpha(B'') = W_1$. Let $w_1 \in W_1$. Since $\eta(\alpha^{-1}(w_1)) = \xi^{-1}\pi^{-1}(\psi^{-1}(w_1) \times g^{-1}(Z'))$, we have that $\alpha^{-1}(w_1)$ is diffeomorphic to $g^{-1}(Z')$ and hence its Euler characteristic $\chi(\alpha^{-1}(w_1))$ is odd. Since $R|_{X_0}$ and η can be chosen arbitrarily close to $j \circ \psi \circ \pi_1 \circ \pi$ in $C^\infty(X_0, W)$ and to the inclusion map $B'' \hookrightarrow T$ in $C^\infty(B'', T)$, respectively, we have that the regular map $R'' : B'' \to W$ defined by $R'' := R \circ \xi|_{B''}$ is arbitrarily close to α in $C^\infty(B'', W)$ also. It follows that $R''(B'') = W_1$ and the fiber $(R'')^{-1}(w_1)$ is diffeomorphic to $\alpha^{-1}(w_1)$ (and hence to $g^{-1}(Z')$). In particular, it holds: $\chi((R'')^{-1}(w_1))$ is odd for each $w_1 \in W_1$, while $\chi((R'')^{-1}(w_2)) = 0$ for each $w_2 \in W_2$. This contradicts Proposition 2.3.2 of [1]. □

Remark 2.1. In the statement of Theorem 1.1, the compact real algebraic manifold X can be chosen connected. Let us explain how to modify the preceding proof in order to obtain such a improvement. Suppose to have just defined the compact connected real algebraic manifold A and the smooth map $\varphi : A \to Y$. Modifying slightly the argument contained in the proof of Lemma 4.7 of [14], we may suppose that the unoriented cobordism class of A is null. It follows that the unoriented bordism class of the map $j \circ \psi \circ \pi_1$ is null (and hence algebraic). Applying

Theorem 2.8.4 of [1] instead of Theorem 2.8.3 of [1] to $j \circ \psi \circ \pi_1$, we see that it is possible to choose $X_0 = X$. In particular, X is diffeomorphic to $S^1 \times A$ and hence it is connected.

Proof of Lemma 1.5. (1) Let Ω be a Zariski open subset of $\mathbb{R}^n$ such that $\Omega \cap Y \neq \emptyset$ and let $\rho : \Omega \to Y$ be a regular map such that $\rho(y) = y$ for each $y \in \Omega \cap Y$. Let $z \in \Omega \cap Y$ and let $T_z(Y)$ be the m-dimensional affine subspace of $\mathbb{R}^n$ tangent to Y at z. It is evident that the restriction of ρ to $\Omega \cap T_z(Y)$ is dominating.

(2) Let $\psi : V \to Z$ be a biregular isomorphism from a nonempty Zariski open subset of $\mathbb{R}^m$ to a Zariski open subset Z of Y. Since ψ^{-1} is regular, by Proposition 2.1.1 of [1], there exist a Zariski open neighborhood Ω' of Z in $\mathbb{R}^n$ and a regular map $f : \Omega' \to \mathbb{R}^m$ which extends ψ^{-1}. Define $\Omega := f^{-1}(V)$ and the regular map $\rho : \Omega \to Y$ by $\rho := \psi \circ f|_{\Omega}$. It follows that Ω is a Zariski open neighborhood of Z in $\mathbb{R}^n$ and $\rho(y) = y$ for each $y \in \Omega \cap Y$. This completes the proof. □

Proof of Theorem 1.8. The first part of the theorem is an immediate consequence of Theorem 1.3 and Lemma 1.5(2). In order to complete the proof, we must only prove that, if $m \geqslant 2$, then $H_1(Y, \mathbb{Z}/2)$ is algebraic. This follows easily from Théorème III.2 of [18], Tognoli's theorem [19] and Theorem 1.1 of [4]. □

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